

The general homothetic equations

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Abstract

In an earlier paper [6] the author wrote the homothetic equations for vacuum solutions in a first order formalism allowing for arbitrary alignment of the dyad. This paper generalises that method to homothetic equations in non-vacuum spaces and also provides useful second integrability conditions. An application to the well-known Petrov type O pure radiation solutions is given.

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1 Recap

In [6] I gave the homothetic Killing equations in vacuum, written out in a first order form without the assumption that the spinor dyad used was aligned to either the symmetry or the curvature in any way, and indeed allowing for the dyad to be non-normalised. In this paper I will generalise to non-vacuum. Conventions and notation will follow Penrose and Rindler [7].

A homothetic vector ξ^a by definition satisfies the equation

$$\xi_{a;b} = F_{ab} + \psi g_{ab}. \quad (1)$$

Here ψ , the **divergence**, is a constant, and F_{ab} will be called the **homothetic bivector**.

Let $\{o^A, \iota^A\}$ be a spinor dyad, with $o_A \iota^A = \chi$. A complex null tetrad is related to this dyad in the standard way:

$$\ell^a = o^A o^{A'}; \quad n^a = \iota^A \iota^{A'}; \quad m^a = o^A \iota^{A'}; \quad \bar{m}^a = \iota^A o^{A'},$$

([7], (4.5.19)), and $\ell_a n^a = -m_a \bar{m}^a = \chi \bar{\chi}$. As in [6], we define components of the homothety:

$$\xi_a = \xi_n \ell_a + \xi_\ell n_a - \xi_{\bar{m}} m_a - \xi_m \bar{m}_a, \quad (2)$$

with $\{\ell^a, n^a, m^a, \bar{m}^a\}$ a Newman-Penrose tetrad. Thus, for example, $\chi \bar{\chi} \xi_\ell = \xi_a \ell^a$.

For the homothetic bivector F_{ab} we define its anti-self dual by

$${}^-F_{ab} = \frac{1}{2} (F_{ab} + i F_{ab}^*) \quad (3)$$

and then

$${}^-F_{ab} = (\chi \bar{\chi})^{-1} (2\phi_{00} \ell_{[a} m_{b]} + 2\phi_{01} (\ell_{[a} n_{b]} - m_{[a} \bar{m}_{b]}) - 2\phi_{11} n_{[a} \bar{m}_{b]}), \quad (4)$$

where

$$\phi_{11} = (\chi\bar{\chi})^{-1} F_{ab} \ell^a m^b \quad (5)$$

$$\phi_{01} = \frac{1}{2} (\chi\bar{\chi})^{-1} (F_{ab} n^a \ell^b - F_{ab} \bar{m}^a m^b) \quad (6)$$

$$\phi_{00} = (\chi\bar{\chi})^{-1} F_{ab} \bar{m}^a n^b \quad (7)$$

Most of the equations in this paper will be given using the compacted GHP-formalism, see [3,7,8]. In this formalism, we concentrate on those spin coefficients of good weight, that is, those that transform homogeneously under a spin-boost transformation of the dyad: if

$$o^A \mapsto \lambda o^A \quad \iota^A \mapsto \mu \iota^A$$

a **weighted** quantity η of **type** $\{r', r; t', t\}$ undergoes a transformation

$$\eta \mapsto \lambda^{r'} \bar{\lambda}^{t'} \mu^r \bar{\mu}^t \eta.$$

These weights will be referred to as the **Penrose-Rindler (PR)** weights, to distinguish them from the more familiar GHP-weights (p, q) for a normalised dyad in e.g. [3,7]. The two sets of weights are related by $p = r' - r$ and $q = t' - t$.

2 General Equations

The Killing equations themselves, (1), are unaffected by the curvature and so are the same as in [6].

$$\mathfrak{p} \xi_\ell = -\bar{\kappa} \xi_m - \kappa \xi_{\bar{m}}; \quad (8a)$$

$$\mathfrak{p}' \xi_\ell = -\bar{\tau} \xi_m - \tau \xi_{\bar{m}} - (\phi_{01} + \bar{\phi}_{01}) + \psi; \quad (8b)$$

$$\mathfrak{d} \xi_\ell = -\bar{\rho} \xi_m - \sigma \xi_{\bar{m}} + \phi_{11}; \quad (8c)$$

$$\mathfrak{p} \xi_n = -\tau' \xi_m - \bar{\tau}' \xi_{\bar{m}} + (\phi_{01} + \bar{\phi}_{01}) + \psi; \quad (8d)$$

$$\mathfrak{p}' \xi_n = -\kappa' \xi_m - \bar{\kappa}' \xi_{\bar{m}}; \quad (8e)$$

$$\mathfrak{d} \xi_n = -\rho' \xi_m - \bar{\sigma}' \xi_{\bar{m}} - \bar{\phi}_{00}; \quad (8f)$$

$$\mathfrak{p} \xi_m = -\bar{\tau}' \xi_\ell - \kappa \xi_n - \phi_{11}; \quad (8g)$$

$$\mathfrak{p}' \xi_m = -\bar{\kappa}' \xi_\ell - \tau \xi_n + \bar{\phi}_{00}; \quad (8h)$$

$$\mathfrak{d} \xi_m = -\bar{\sigma}' \xi_\ell - \sigma \xi_n; \quad (8i)$$

$$\mathfrak{d}' \xi_m = -\bar{\rho}' \xi_\ell - \rho \xi_n + (\phi_{01} - \bar{\phi}_{01}) - \psi; \quad (8j)$$

The spin-boost weights (r', r, t', t) of the components of ξ^a and F_{ab} are given in Table I (correcting a minor typo in [6]).

The Ricci identity for ξ^a implies $F_{cd;b} = R_{abcd} \xi^a$, from which the algebraic Bianchi identities lead to equations for the derivatives of the ϕ_{ij} . The anti-self-dual of the above equation takes the spinor form

$$\nabla_{CC'} \phi_{AB} = (\Psi_{ABDC} \epsilon_{D'C'} + \Phi_{ABD'C'} \epsilon_{DC}) \xi^{DD'} - \Lambda (\epsilon_{BC} \xi_{AC'} + \epsilon_{AC} \xi_{BC'}) \quad (9)$$

	ξ_ℓ	ξ_n	ξ_m	$\xi_{\bar{m}}$	ϕ_{00}	ϕ_{01}	ϕ_{11}
r'	0	-1	1	0	-1	0	1
r	-1	0	-1	0	1	0	-1
t'	0	-1	-1	0	0	0	0
t	-1	0	0	-1	0	0	0

Table 1: weights of components

Here Ψ_{ABCD} is the (totally symmetric) Weyl spinor, $\Phi_{ABA'B'}$ the Ricci spinor and $24\Lambda = R$, the Ricci scalar (see [7]).

The components of the Weyl and Ricci spinors are given in [7] (4.11.6) and (4.11.8) respectively, and then resolving equation (9) we get the (first) integrability conditions

$$\Psi_1\xi_\ell - \Psi_0\xi_{\bar{m}} + \Phi_{01}\xi_\ell - \Phi_{00}\xi_m = 2\kappa\phi_{01} - \mathfrak{p}\phi_{11}; \quad (10a)$$

$$\Psi_1\xi_m - \Psi_0\xi_n + \Phi_{02}\xi_\ell - \Phi_{01}\xi_m = 2\sigma\phi_{01} - \mathfrak{d}\phi_{11}; \quad (10b)$$

$$\Psi_2\xi_\ell - \Psi_1\xi_{\bar{m}} + \Phi_{01}\xi_{\bar{m}} - \Phi_{00}\xi_n + 2\Pi\xi_\ell = 2\rho\phi_{01} - \mathfrak{d}'\phi_{11}; \quad (10c)$$

$$\Psi_2\xi_m - \Psi_1\xi_n + \Phi_{02}\xi_{\bar{m}} - \Phi_{01}\xi_n + 2\Pi\xi_m = 2\tau\phi_{01} - \mathfrak{p}'\phi_{11}; \quad (10d)$$

$$\Psi_3\xi_\ell - \Psi_2\xi_{\bar{m}} + \Phi_{21}\xi_\ell - \Phi_{20}\xi_m - 2\Pi\xi_{\bar{m}} = 2\tau'\phi_{01} - \mathfrak{p}\phi_{00}; \quad (10e)$$

$$\Psi_3\xi_m - \Psi_2\xi_n + \Phi_{22}\xi_\ell - \Phi_{21}\xi_m - 2\Pi\xi_n = 2\rho'\phi_{01} - \mathfrak{d}\phi_{00}; \quad (10f)$$

$$\Psi_4\xi_\ell - \Psi_3\xi_{\bar{m}} + \Phi_{21}\xi_{\bar{m}} - \Phi_{20}\xi_n = 2\sigma'\phi_{01} - \mathfrak{d}'\phi_{00}; \quad (10g)$$

$$\Psi_4\xi_m - \Psi_3\xi_n + \Phi_{22}\xi_{\bar{m}} - \Phi_{21}\xi_n = 2\kappa'\phi_{01} - \mathfrak{p}'\phi_{00} \quad (10h)$$

$$\Psi_2\xi_\ell - \Psi_1\xi_{\bar{m}} + \Phi_{11}\xi_\ell - \Phi_{10}\xi_m - \Pi\xi_\ell = \mathfrak{p}\phi_{01} - \tau'\phi_{11} - \kappa\phi_{00}; \quad (10i)$$

$$\Psi_2\xi_m - \Psi_1\xi_n + \Phi_{12}\xi_\ell - \Phi_{11}\xi_m - \Pi\xi_m = \mathfrak{d}\phi_{01} - \rho'\phi_{11} - \sigma\phi_{00}; \quad (10j)$$

$$\Psi_3\xi_\ell - \Psi_2\xi_{\bar{m}} + \Phi_{11}\xi_{\bar{m}} - \Phi_{10}\xi_n + \Pi\xi_{\bar{m}} = \mathfrak{d}'\phi_{01} - \rho\phi_{00} - \sigma'\phi_{11}; \quad (10k)$$

$$\Psi_3\xi_m - \Psi_2\xi_n + \Phi_{12}\xi_{\bar{m}} - \Phi_{11}\xi_n + \Pi\xi_n = \mathfrak{p}'\phi_{01} - \tau\phi_{00} - \kappa'\phi_{11}. \quad (10l)$$

where $\Pi = \chi\bar{\chi}\Lambda$. These are equivalent to the equations (20)–(22) in [5].

Note that there are four pairs of equations with the same Weyl curvature terms (c/i ; d/j ; e/k and f/l). We can eliminate the Weyl curvature terms between these pairs to give

equations equivalent to (23) in [5]:

$$\begin{aligned} \mathfrak{p} \phi_{01} + \delta' \phi_{11} - \kappa \phi_{00} - 2\rho \phi_{01} - \tau' \phi_{11} \\ = (\Phi_{11} - 3\Lambda) \xi_\ell + \Phi_{00} \xi_n - \Phi_{10} \xi_m - \Phi_{01} \xi_{\overline{m}} \end{aligned} \quad (11a)$$

$$\begin{aligned} \delta \phi_{01} + \mathfrak{p}' \phi_{11} - \sigma \phi_{00} - 2\tau \phi_{01} - \rho' \phi_{11} \\ = \Phi_{12} \xi_\ell + \Phi_{01} \xi_n - (\Phi_{11} + 3\Lambda) \xi_m - \Phi_{02} \xi_{\overline{m}} \end{aligned} \quad (11b)$$

$$\begin{aligned} \delta' \phi_{01} + \mathfrak{p} \phi_{00} - \rho \phi_{00} - 2\tau' \phi_{01} - \sigma' \phi_{11} \\ = -\Phi_{21} \xi_\ell - \Phi_{10} \xi_n + \Phi_{20} \xi_m + (\Phi_{11} + 3\Lambda) \xi_{\overline{m}} \end{aligned} \quad (11c)$$

$$\begin{aligned} \mathfrak{p}' \phi_{01} + \delta \phi_{00} - \tau \phi_{00} - 2\rho' \phi_{01} - \kappa' \phi_{11} \\ = -\Phi_{22} \xi_\ell - (\Phi_{11} - 3\Lambda) \xi_n + \Phi_{21} \xi_m + \Phi_{12} \xi_{\overline{m}} \end{aligned} \quad (11d)$$

Note that all these equations are consistent as far as spin and boost weight are concerned, and all reduce to the equations of [6] in vacuum.

3 Second integrability conditions

Since a homothetic transformation preserves connection and hence curvature, we have $\mathcal{L}_\xi R^a_{bcd} = 0$, and resolving the spinor version of this equation and using (10) to eliminate first derivatives of the ϕ_{ij} leads to equations I will refer to as second integrability conditions, although they are not strictly integrability conditions in the case of a homothety. The same equations arises from applying the commutators to the components of the homothetic bivector of course. Using the Bianchi identities and the GHP-notation these equations can be reduced to a very compact form. Firstly, define the zero weight derivative operator

$$\mathcal{L}_\xi = \xi_n \mathfrak{p} + \xi_\ell \mathfrak{p}' - \xi_m \delta' - \xi_{\overline{m}} \delta,$$

and let

$$X_{00} = \phi_{00} - \kappa' \xi_\ell - \tau' \xi_n + \sigma' \xi_m + \rho' \xi_{\overline{m}} \quad X_{11} = \phi_{11} + \kappa \xi_n + \tau \xi_\ell - \sigma \xi_{\overline{m}} - \rho \xi_m.$$

(Note that under the Sachs $*$ operation, X_{11} and X_{00} are unchanged but $\overline{X}_{11}^* = \overline{X}_{00}$ and $\overline{X}_{00}^* = \overline{X}_{11}$). Then we find that

$$\mathcal{L}_\xi \Psi_i + 2\psi \Psi_i = iX_{00} \Psi_{i-1} - p \phi_{01} \Psi_i + (i-4)X_{11} \Psi_{i+1} \quad (12)$$

$$\begin{aligned} \mathcal{L}_\xi \Phi_{ab} + 2\psi \Phi_{ab} = aX_{00} \Phi_{(a-1)b} + b\overline{X}_{00} \Phi_{a(b-1)} - (p \phi_{01} + q \overline{\phi}_{01}) \Phi_{ab} \\ + (a-2)X_{11} \Phi_{(a+1)b} + (b-2)\overline{X}_{11} \Phi_{a(b+1)} \end{aligned} \quad (13)$$

$$\mathcal{L}_\xi \Pi + 2\psi \Pi = 0, \quad (14)$$

where $p = r' - r$ and $q = t' - t$ are the GHP weights [7,8]. Note that Ψ_i has PR-weight $[3-i, i-1, 1, 1]$ and Φ_{ab} PR-weight $[2-a, a, 2-b, b]$. Equations (12) are equivalent to Collinson and French's equations (2.2) [1] and Kolassis and Ludwig's equations (43)–(45)

[4]; equations (13) are equivalent to [4] equations (47)–(49). In these references the tetrad is assumed normalised. A comparison with [4] shows that equations (12) are the same for a general conformal vector in a normalised tetrad: this is as expected since these equations actually arise from the derivative of the Weyl tensor part of the curvature.

Note that if ℓ^a is a Debever-Penrose direction then $\Psi_0 = 0$ and (12) implies $X_{11} = 0$, or

$$\phi_{11} = -\kappa\xi_n - \tau\xi_\ell + \sigma\xi_{\bar{m}} + \rho\xi_m \quad (15)$$

correcting the error in equation (11) of [6]. Similarly, if n^a is a Debever-Penrose direction then $X_{00} = 0$.

4 Type O pure radiation metrics

Possibly the simplest non vacuum metric to consider would be that of conformally flat pure radiation solutions. In [2] Edgar and Ludwig performed the integration of this case, which I here repeat with the extra assumption of the existence of a homothetic vector, when the calculations can be pushed to completion in the sense that no free functions remain. The main difference here is that I use the homothety to choose coordinate candidates.

We begin by assuming that the dyad is normalised ($\chi = 1$) and aligned to the Ricci tensor, so $R_{ab} = \Phi_{22}\ell_a\ell_b$, leaving complete four parameter null rotation freedom in choosing the dyad.

The Bianchi identities quickly tell us that $\kappa = \sigma = 0$, and the remaining Bianchi identities (see [7]) are

$$\delta'\Phi_{22} = \bar{\tau}\Phi_{22}, \quad \mathfrak{p}\Phi_{22} = \bar{\rho}\Phi_{22} \quad \mathfrak{p}\Phi_{22} = (\rho + \bar{\rho})\Phi_{22}$$

So $\rho = 0$ and we are in Kundt's class. Now if $\tau = 0$ we have plane waves, a case which has been much studied and we will ignore. So from henceforth, $\tau \neq 0$.

Of the second integrability equations only equation (13) for (a, b) either $(1, 2)$ or $(2, 2)$ are non-trivial, and the first of these gives $\phi_{11} = -\tau\xi_\ell$. Suppose ϕ_{11} , and hence ξ_ℓ , vanishes. Then equation (10d) and $\tau \neq 0$ implies ϕ_{01} vanishes and then by (10l), $\phi_{00} = 0$. But now (10d) gives $\xi_{\bar{m}} = 0$, and the Killing equation (8h) means that $\xi_n = 0$ and the homothetic vector vanishes.

Thus ξ_ℓ is not identically zero: no symmetry vector is orthogonal to ℓ^a . This means $\phi_{11} \neq 0$ and we can perform a proper null rotation about ℓ^a to set ϕ_{01} to be identically zero: we will be left with boost and rotation freedom, which is what we would want for a GHP integration procedure.

With $\phi_{01} = 0$, the integrability equations involving the derivatives of ϕ_{01} , (10i) – (10l) show that $\sigma' = \rho' = \tau' = 0$ and $\phi_{00} = \kappa'\xi_\ell$.

The remaining integrability conditions (10) and (13) allow us to find all directional

derivatives of all the remaining scalars. We have

$$\begin{array}{lll}
\mathfrak{p} \tau = 0 & \mathfrak{p} \kappa' = 0 & \mathfrak{p} \Phi_{22} = 0 \\
\mathfrak{d} \tau = \tau^2 & \mathfrak{d} \kappa' = \tau \kappa' - \Phi_{22} & \mathfrak{d} \Phi_{22} = \tau \Phi_{22} \\
\mathfrak{d}' \tau = \tau \bar{\tau} & \mathfrak{d}' \kappa' = \kappa' \bar{\tau} & \mathfrak{d}' \Phi_{22} = \bar{\tau} \Phi_{22} \\
\xi_\ell \mathfrak{p}' \tau = \tau(W - \psi) & \xi_\ell \mathfrak{p}' \kappa' = -\xi_{\bar{m}} \Phi_{22} + \kappa'(W - \psi) & \xi_\ell \mathfrak{p}' \Phi_{22} = \Phi_{22}(W - 2\psi)
\end{array}$$

where $W = \bar{\tau} \xi_m + \tau \xi_{\bar{m}}$ has weight $(0, 0)$. The homothetic equations reduce to

$$\begin{array}{lll}
\mathfrak{p} \xi_\ell = 0 & \mathfrak{p} \xi_n = \psi & \mathfrak{p} \xi_m = \tau \xi_\ell \\
\mathfrak{d} \xi_\ell = -\tau \xi_\ell & \mathfrak{d} \xi_n = -\bar{\kappa}' \xi_\ell & \mathfrak{d} \xi_m = 0 \\
\mathfrak{d}' \xi_\ell = -\bar{\tau} \xi_\ell & \mathfrak{d}' \xi_n = -\kappa' \xi_\ell & \mathfrak{d}' \xi_m = -\psi \\
\mathfrak{p}' \xi_\ell = -W + \psi & \mathfrak{p}' \xi_n = -\kappa' \xi_m - \bar{\kappa}' \xi_{\bar{m}} & \mathfrak{p}' \xi_m = -\tau \xi_n
\end{array}$$

In [2], Edgar and Ludwig used the tetrad freedom to set $\tau' = \sigma' = \rho' = 0$, $\Phi_{22} - \tau \kappa' - \bar{\tau} \bar{\kappa}' = 0$ and $\mathfrak{p}'(\tau/\bar{\tau}) = 0$ as a preparation to performing the integration. In our approach, we have obtained $\tau' = \sigma' = \rho' = 0$ without the need to solve a system of differential equations, and it is not difficult to check that we also have $\mathfrak{p}'(\tau/\bar{\tau}) = 0$. As for Edgar and Ludwig's other term, we note that $\Phi_{22} - \tau \kappa' - \bar{\tau} \bar{\kappa}'$ is of weight $(-2, -2)$. Define the real scalar $Z = \xi_\ell^2(\Phi_{22} - \tau \kappa' - \bar{\tau} \bar{\kappa}')$ of weight $(0, 0)$. Then we can easily check that Z is annihilated by all the derivative operators and is hence constant.

We will not attempt to show Z is zero, as we wish to pick a set of coordinate candidates (real weight $(0, 0)$ scalars) more attuned to the homothetic or Killing vector than those of Edgar and Ludwig.

Firstly, as in [2], define the convenient scalars $P = \sqrt{\tau/2\bar{\tau}}$, complex of weight $(1, -1)$ and $A = (2\tau\bar{\tau})^{-1/2}$, real of weight $(0, 0)$. The scalar P has the happy property of being annihilated by all the GHP operators, whereas

$$\mathfrak{p} A = 0, \quad \mathfrak{p}' A = \frac{1}{\xi_\ell}(A\psi - u), \quad \mathfrak{d} A = -P. \quad (16)$$

Our coordinate candidates are w , x and y where

$$w = \Re\left(\frac{\xi_m}{P}\right), \quad z = x + iy = -2\frac{\kappa' P}{\Phi_{22}},$$

and also a real scalar u of weight $(0, 0)$ satisfying the equations (cf. [2])

$$\mathfrak{p} u = 0, \quad \mathfrak{d} u = 0, \quad \mathfrak{p}' u = 1/\xi_\ell.$$

These latter equations are consistant, as can be checked by verifying the commutators are satisfied. The commutators acting on weight $(0, 0)$ scalars here simplify to

$$[\mathfrak{p}, \mathfrak{p}'] = A^{-1} \left(\frac{1}{P} \mathfrak{d} + \frac{1}{P} \mathfrak{d}' \right), \quad [\mathfrak{p}, \mathfrak{d}] = [\mathfrak{d}, \mathfrak{d}'] = 0, \quad [\mathfrak{p}', \mathfrak{d}] = -\frac{1}{AP} \mathfrak{p}' - \kappa' \mathfrak{p}$$

and their conjugates.

Our table of derivatives reads

$$\begin{array}{lll}
\flat u = 0 & \flat' u = 1/\xi_\ell & \flat u = 0 \\
\flat w = \xi_\ell/A & \flat' w = -B/\xi_\ell & \flat w = -\psi P \\
\flat x = 0 & \flat' x = \frac{1}{\xi_\ell}(\psi x + w) & \flat x = P \\
\flat y = 0 & \flat' y = -\frac{1}{\xi_\ell}(\psi y - v) & \flat y = -iP
\end{array}$$

where $B = \xi_n \xi_\ell / A$ and $v = \Im(\xi_m / P)$ are both real of weight $(0, 0)$. With the coordinates in the order (u, w, x, y) , the tetrad is thus

$$\ell^a = \xi_\ell (0, A^{-1}, 0, 0) \quad n^a = \xi_\ell^{-1} (1, -B, \psi x + w, \psi y - v,) \quad m^a = P (0, -\psi, 1, -i)$$

The ξ_ℓ and P terms are a manifestation of the residual boost and spin freedom of course.

The other homothetic vector components are $\xi_n = AB/\xi_\ell$ and $\xi_m = (w + iv)P$, and the homothetic vector simplifies to

$$\xi^a = (1, \psi w, \psi x, \psi y)$$

so that in the Killing case u is a cyclic coordinate.

To complete the integration we need to find A , B and v . Since they are all weight $(0, 0)$ this is straightforward:

$$v = \psi y - c_3 e^{\psi u}, \quad A = c_2 e^{\psi u} - x, \quad B = \psi w + \psi^2 x + (x^2 + y^2) c_1 e^{-\psi u} + c_4 e^{\psi u},$$

where the c_i are constant. Note that in the Killing case, v is constant.

The metric is then

$$ds^2 = H du^2 + 2A du dw + 2(w + c_2 \psi e^{\psi u}) du dx + 2c_3 e^{\psi u} du dy - dx^2 - dy^2$$

where

$$H = 2A (c_1 (x^2 + y^2) e^{-\psi u} + c_4 e^{\psi u}) - c_3^2 e^{2\psi u} - w^2 - \psi^2 x^2 - 2\psi wx.$$

4.1 Larger algebra

This leaves us to consider the solutions with further homothetic or Killing vectors. Since the bracket of two homothetic vectors is Killing, we begin by assuming the first symmetry vector is Killing, and then look for a second (possibly homothetic) vector ξ^a . So the metric is

$$ds^2 = H du^2 + 2(c_2 - x) du dw + 2w du dx + 2c_3 du dy - dx^2 - dy^2$$

where

$$H = 2 ((x^2 + y^2) c_2 + c_4) (c_2 - x) - w^2 - c_3^2$$

and the Killing vector is $K^a = \partial_u$. For the tetrad we perform a rotation to make τ (and hence P) real and also a boost to allow us to integrate more easily: so

$$\ell^a = (0, 1, 0, 0) \quad n^a = \frac{1}{c_2 - x} (1, -c_4 - c_1(x^2 + y^2), w, c_3) \quad m^a = \frac{1}{\sqrt{2}} (0, 0, 1, -i).$$

The advantage of this tetrad is that the improperly weighted spin coefficients ε , α , β and γ all vanish, leaving only

$$\tau = \frac{1}{\sqrt{2}(c_2 - x)} \quad \text{and} \quad \kappa' = -\sqrt{2}c_1 \frac{x + iy}{c_2 - x}.$$

So the four GHP operators reduce to the basic Newman-Penrose operators D , δ , δ' and D' (see [7,8]) for all scalars. Furthermore, the same argument as used in the previous section tells us that ξ_ℓ is non-zero and $\phi_{11} = -\tau\xi_\ell$.

Integrating (8a) and (8b) and using the reality of ξ_ℓ gives $\xi_\ell = F_1(u)(c_2 - x)$ for some $F_1(u)$. Next we integrate (8g), (8i) and (8j) to get

$$\xi_m = \frac{1}{\sqrt{2}} (F_1(u)w - \psi(x - iy)) + F_2(u)$$

for some complex $F_2(u)$. However, (10d) and the Killing equations imply that $F_2(u) = \frac{1}{\sqrt{2}}c_2\psi + ib(u)$ for real b . And now equation (13) for $(a, b) = (2, 2)$ gives $\phi_{01} = \frac{1}{4}\psi$ and so (10l) gives $\phi_{00} = \kappa'\xi_\ell$.

The remaining equation for ξ_ℓ , (8b), shows that $F_1 = a_1 - \frac{1}{2}\psi u$ where a_1 is a constant, so $\xi_\ell = \xi^a \ell_a = (a_1 - \frac{1}{2}\psi u)(c_2 - x)$. But $K^a \ell_a = c_2 - x$, which tells us that we cannot have a second Killing vector ($\psi = 0$), as then a linear combination of ξ^a and K^a would be orthogonal to ℓ^a , which we saw is not possible. Hence

Theorem 1 *Type O pure radiation metrics can admit at most a 2 parameter group of homothetic motions, and if the dimension is 2 it is a proper homothetic group.*

We also see that for the second symmetry vector (the proper homothety), we can take

$$\xi_\ell = -\frac{1}{2}\psi u(c_2 - x).$$

The imaginary part of (8h) can now be solved for $b_2(u)$ to give $b_2(u) = -\frac{1}{\sqrt{2}}\psi c_3 u + a_2$ for constant a_2 . The real part of (8h) now gives us

$$\xi_n = -\frac{1}{2}\psi (c_1(x^2 + y^2)u + c_4 u - w).$$

The only remaining Killing or integrability equation is (8e) for $D'\xi_n$, and this tells us that $c_2 = c_3 = c_4 = a_2 = 0$ and thus

$$\xi^a = \frac{1}{2}\psi (-u, 3w, x, y).$$

The remaining constant c_1 is non-zero (or the metric is flat) and can be absorbed in the coordinates. The type O pure radiation solution with the largest possible homothetic symmetry group is thus

$$ds^2 = - (2x(x^2 + y^2) + w^2) du^2 - 2x du dw + 2w du dx - dx^2 - dy^2$$

with a non-abelian homothetic algebra generated by $\{\partial_u, -u\partial_u + 3w\partial_w + x\partial_x + y\partial_y\}$.

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